

Recursion, Induction, and Other Demons

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Recursive Definitions in PVS

Suppose we want to define a function to sum the first n natural numbers:

$$\text{sum}(n) = \sum_{i=0}^n i.$$

In PVS:

```
sum(n): RECURSIVE nat =  
  IF n = 0 THEN 0 ELSE n + sum(n - 1) ENDIF  
  MEASURE n
```

Functions in PVS are Total

Two Type Correctness Conditions(TCCs):

- ▶ The argument for the recursive call is a natural number.

```
% Subtype TCC generated for n - 1
% expected type nat
sum_TCC1: OBLIGATION FORALL (n: nat):
  NOT n = 0 IMPLIES n - 1 >= 0;
```

- ▶ The recursion terminates.

```
% Termination TCC generated for sum(n - 1)
sum_TCC2: OBLIGATION FORALL (n: nat):
  NOT n = 0 IMPLIES n - 1 < n;
```

A Simple Property of Sum

We would like to prove the following closed form solution to `sum`:

$$\sum_{i=0}^n i = \frac{n(n+1)}{2}.$$

In PVS:

```
closed_form: THEOREM
  sum(n) = (n * (n + 1)) / 2
```

Induction Proofs

`(induct/$ var &optional (fnum 1) name) :`

Selects an induction scheme according to the type of VAR in FNUM and uses formula FNUM to formulate an induction predicate, then simplifies yielding base and induction cases. The induction scheme can be explicitly supplied as the optional NAME argument.

Induction Schemes from the Prelude

```
% Weak induction on naturals.
nat_induction: LEMMA
  (p(0) AND (FORALL j: p(j) IMPLIES p(j+1)))
    IMPLIES (FORALL i: p(i))

% Strong induction on naturals.
NAT_induction: LEMMA
  (FORALL j: (FORALL k: k < j IMPLIES p(k)) IMPLIES p(j))
    IMPLIES (FORALL i: p(i))
```

Proof by Induction

```
closed_form :
```

```
  |-----
```

```
{1}  FORALL (n: nat): sum(n) = (n * (n + 1)) / 2
```

```
Rule? (induct "n")
```

```
Inducting on n on formula 1,
```

```
this yields 2 subgoals:
```


Base Case

```
closed_form.1 :
```

```
  |-----  
{1}  sum(0) = (0 * (0 + 1)) / 2
```

Rule? (grind)

Rewriting with sum

Trying repeated skolemization, instantiation, and if-lifting,

This completes the proof of closed_form.1.

Induction Step

closed_form.2 :

```

|-----
{1}  FORALL j:
      sum(j) = (j * (j + 1)) / 2 IMPLIES
      sum(j + 1) = ((j + 1) * (j + 1 + 1)) / 2

```

Rule? (skosimp*)

Repeatedly Skolemizing and flattening, this simplifies to:

closed_form.2 :

```

{-1}  sum(j!1) = (j!1 * (j!1 + 1)) / 2
|-----
{1}  sum(j!1 + 1) = ((j!1 + 1) * (j!1 + 1 + 1)) / 2

```

Rule? (undo)y

closed_form.2 :

```

|-----
{1}  FORALL j:
      sum(j) = (j * (j + 1)) / 2 IMPLIES
      sum(j + 1) = ((j + 1) * (j + 1 + 1)) / 2

```

Rule? (skip)

Skolemizing with the names of the bound variables,
this simplifies to:

closed_form.2 :

```

{-1} sum(j) = (j * (j + 1)) / 2
|-----
{1}  sum(j + 1) = ((j + 1) * (j + 1 + 1)) / 2

```

$$\{-1\} \quad \text{sum}(j) = (j * (j + 1)) / 2$$

$$\{1\} \quad \text{sum}(j + 1) = ((j + 1) * (j + 1 + 1)) / 2$$

Rule? (expand "sum" +)

Expanding the definition of sum,
this simplifies to:

closed_form.2 :

$$\{-1\} \quad \text{sum}(j) = (j * (j + 1)) / 2$$

$$\{1\} \quad 1 + \text{sum}(j) + j = (2 + j + (j * j + 2 * j)) / 2$$

$$[-1] \quad \text{sum}(j) = (j * (j + 1)) / 2$$

|-----

$$\{1\} \quad 1 + \text{sum}(j) + j = (2 + j + (j * j + 2 * j)) / 2$$

Rule? (assert)

Simplifying, rewriting, and recording with decision procedures,

This completes the proof of closed_form.2.

Q.E.D.

Automated Simple Induction Proofs

```
|-----
{1}  FORALL (n: nat): sum(n) = (n * (n + 1)) / 2
```

Rule? (`induct-and-simplify "n"`)

Rewriting with sum

Rewriting with sum

By induction on n, and by repeatedly rewriting and simplifying,

Q.E.D.

Limitations of automation

Consider the n th factorial:

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n(n-1)!, & \text{otherwise.} \end{cases}$$

In PVS:

```
factorial(n : nat): RECURSIVE posnat =  
  IF n = 0 THEN 1 ELSE n * factorial(n - 1) ENDIF  
MEASURE n
```

A Simple Property of Factorial

$$\forall x > 3 : x! > 2^x$$

In PVS:

```
x: var above(3)
```

```
factorial_gt_expt2_above3: LEMMA  
  factorial(x) > 2 ^ x
```


A Series of Unfortunate Events ...

```
factorial_gt_expt2_above3 :
```

```
  |-----
```

```
{1}  FORALL (x: above(3)): factorial(x) > 2 ^ x
```

```
Rule? (induct-and-simplify "x")
```

```
Rewriting with factorial
```

```
Rewriting with factorial
```

```
Rewriting with factorial
```

```
...
```

Whenever the theorem prover falls into an infinite loop, the Emacs command `C-c C-c` will force PVS to break into Lisp. The Lisp command `(restore)` will return to the PVS state prior to the last proof command.

...

Rewriting with factorial

Error: Received signal number 2 (Interrupt)

[condition type: interrupt-signal]

...

[1c] pvs(87): (restore)

factorial_gt_expt2_above3 :

|-----

{1} FORALL (x: above(3)): factorial(x) > 2 ^ x

Rule?

Whenever the theorem prover falls into an infinite loop, the Emacs command `C-c C-c` will force PVS to break into Lisp. The Lisp command `(restore)` will return to the PVS state prior to the last proof command.

...

Rewriting with factorial

Error: Received signal number 2 (Interrupt)

[condition type: interrupt-signal]

...

[1c] pvs(87): (restore)

factorial_gt_expt2_above3 :

|-----

{1} FORALL (x: above(3)): factorial(x) > 2 ^ x

Rule?

Exercise 1

Prove this formula in PVS:

$$\forall x > 3 : x! > 2^x$$

Hint: See Exercises/Lecture-2.pvs.

Induction-Free Induction

Consider a common implementation of the n -th factorial in an imperative programming language:

```
/* Pre: n >= 0 */  
  
int a = 1;  
for (int i=0;i < n;i++) {  
    /* Inv: a = i! */  
    a = a*(i+1);  
}  
  
/* Post: a = n! */
```

In PVS ...

```
fact_it(n:nat,i:upto(n),a:posnat) : RECURSIVE posnat =  
  IF   i = n THEN a  
  ELSE fact_it(n,i+1,a*(i+1))  
  ENDIF  
MEASURE n-i
```

```
fact_it_correctness : THEOREM  
  fact_it(n,0,1) = factorial(n)
```

Proving `fact_it_correctness`

```
|-----
{1}  FORALL (n: nat): fact_it(n, 0, 1) = factorial(n)
```

Rule? (`induct-and-simplify "n"`)

this simplifies to:

```
fact_it_correctness :
```

```
{-1} fact_it(j!1, 0, 1) = factorial(j!1)
```

```
|-----
{1}  fact_it(1 + j!1, 1, 1) =
      factorial(j!1) + factorial(j!1) * j!1
```

The proof by (explicit) induction requires an inductive proof of an auxiliary lemma.

Induction-Free Induction By Predicate Subtyping

```
fact_it(n:nat, i:upto(n), (a:posnat | a=factorial(i))) :  
  RECURSIVE {b:posnat | b=factorial(n)} =  
  IF   i = n THEN a  
  ELSE fact_it(n, i+1, a*(i+1))  
  ENDIF  
MEASURE n-i  
  
n : VAR nat  
  
fact_it_correctness : LEMMA  
  fact_it(n, 0, 1) = factorial(n)  
%|- fact_t_correctness : PROOF (skeep) (assert) QED
```


There is No Free Lunch

```

fact_it_TCC4 :
  |-----
  {1}  FORALL (n: nat, i: upto(n),
            (a: nat | a = factorial(i))):
        NOT i = n IMPLIES a * (i + 1) = factorial(1 + i)

```

Rule? (skeep :preds? t)

```

fact_it_TCC4 :
{-1}  n >= 0
{-2}  i <= n
{-3}  a = factorial(i)
  |-----
  {1}  i = n
  {2}  a * (i + 1) = factorial(1 + i)

```

Rule? (expand "factorial" 2)

fact_it_TCC4 :

[-1] n >= 0

[-2] i <= n

[-3] a = factorial(i)

|-----

[1] i = n

{2} a * i + a = factorial(i) + factorial(i) * i

Rule? (assert)

Q.E.D.

You Can Also Pay at the Exit

```
fact_it2(n:nat, i:upto(n), a:posnat) : RECURSIVE
  {b:posnat | b = a*factorial(n)/factorial(i)} =
  IF   i = n THEN a
  ELSE fact_it2(n, i+1, a*(i+1))
  ENDIF
MEASURE n-i
```

```
fact_it2_correctness : LEMMA
  fact_it2(n, 0, 1) = factorial(n)
```

```
|-----
{1}  FORALL (n: nat): fact_it2(n, 0, 1) = factorial(n)
```

Rule? (skeep)

```
|-----
{1}  fact_it2(n, 0, 1) = factorial(n)
```

Rule? (typepred "fact_it2(n,0,1)")

```
{-1} fact_it2(n, 0, 1) > 0
{-2} fact_it2(n, 0, 1) = 1 * factorial(n) / factorial(0)
|-----
[1]  fact_it2(n, 0, 1) = factorial(n)
```

Rule? (expand "factorial" -2 2)

[-1] fact_it2(n, 0, 1) > 0

{-2} fact_it2(n, 0, 1) = 1 * factorial(n) / 1

|-----

[1] fact_it2(n, 0, 1) = factorial(n)

Rule? (assert)

Q.E.D.

But The Price is Higher

```

fact_it2_TCC5: OBLIGATION
  FORALL (n: nat, i: upto(n),
    v:
      [d1: z: [n: nat, upto(n), posnat] |
        z'1 - z'2 < n - i ->
        b: posnat | b = d1'3 * factorial(d1'1) /
          factorial(d1'2)],
    a: posnat):
  NOT i = n IMPLIES
    v(n, i + 1, a * (i + 1)) =
      a * factorial(n) / factorial(i);

```

Rule? (skeep :preds? t)

Skolemizing with the names of the bound variables,
this simplifies to:

fact_it2_TCC5 :

{-1} n >= 0

{-2} i <= n

{-3} a > 0

|-----

{1} i = n

{2} $v(n, i + 1, a * (i + 1)) = a * \text{factorial}(n) /$
 $\text{factorial}(i)$

Rule? (name-replace "HI" "v(n, i + 1, a * (i + 1))")
 Using HI to name and replace v(n, i + 1, a * (i + 1)),
 this yields 2 subgoals:
 fact_it2_TCC5.1 :

```

[-1]  n >= 0
[-2]  i <= n
[-3]  a > 0
      |-----
[1]   i = n
{2}   HI = a * factorial(n) / factorial(i)
  
```


Rule? (typepred "HI")

Adding type constraints for HI,

this simplifies to:

fact_it2_TCC5.1 :

{-1} HI > 0

{-2} $HI = \frac{\text{factorial}(n) * a + \text{factorial}(n) * a * i}{\text{factorial}(1 + i)}$

[-3] n >= 0

[-4] i <= n

[-5] a > 0

|-----

[1] i = n

[2] $HI = a * \text{factorial}(n) / \text{factorial}(i)$

Rule? (expand "factorial" -2 3)

Expanding the definition of factorial,
this simplifies to:

fact_it2_TCC5.1 :

[-1] HI > 0

{-2} HI =

$$\frac{\text{factorial}(n) * a + \text{factorial}(n) * a * i}{\text{factorial}(i) + \text{factorial}(i) * i}$$

[-3] n >= 0

[-4] i <= n

[-5] a > 0

|-----

[1] i = n

[2] HI = a * factorial(n) / factorial(i)

Rule? (replaces -2)

Iterating REPLACE,
this simplifies to:
fact_it2_TCC5.1 :

```

{-1}  (factorial(n) * a + factorial(n) * a * i) /
      (factorial(i) + factorial(i) * i)
      > 0
{-2}  n >= 0
{-3}  i <= n
{-4}  a > 0
      |-----
{1}   i = n
{2}   (factorial(n) * a + factorial(n) * a * i) /
      (factorial(i) + factorial(i) * i)
      = a * factorial(n) / factorial(i)

```

Rule? (grind-reals)

Rewriting with `pos_div_gt`

Rewriting with `cross_mult`

Applying GRIND-REALS,

This completes the proof of `fact_it2_TCC5.1`.

- ▶ All the other subgoals are discharged by `(assert)`.

Induction-Free Induction

- + Induction scheme based the recursive definition of the function not on the measure function!.
- + Proofs exploit type-checker power.
 - Some TCCs look scary (but they are easy to tame)
 - If you modify the definitions, the TCCs get re-arranged (be careful or you can lose your proof)
- ? Can this method be used when the recursive function was not originally typed that way?

Recursive Judgments

Consider the Ackermann function:

$$A(m,n) = \begin{cases} n + 1, & \text{if } m = 0 \\ A(m - 1, 1), & \text{if } m > 0 \text{ and } n = 0 \\ A(m - 1, A(m, n - 1)), & \text{otherwise.} \end{cases}$$

In PVS:

```
ack(m,n) : RECURSIVE nat =
  IF      m = 0 THEN n+1
  ELSIF  n = 0 THEN ack(m-1,1)
  ELSE   ack(m-1,ack(m,n-1))
  ENDIF
MEASURE ?lex2(m,n)
```

Exercise 2

Prove:

$$\forall m, n : A(m, n) > m + n$$

In PVS:

```
ack_simple_property : THEOREM
  FORALL(m,n): ack(m,n) > m+n
```

Hint: You may need two inductions!

Recursive Judgements

```
ack_gt_m_n : RECURSIVE JUDGEMENT
  ack(m,n) HAS_TYPE above(m+n)
```

The type checker generates TCCs corresponding to the recursive definition of the type-restricted version of `ack`, e.g.,

```
ack_gt_m_n_TCC1: OBLIGATION FORALL (m, n: nat): m=0 IMPLIES
  n+1 > m+n;
```

```
ack_gt_m_n_TCC3: OBLIGATION
  FORALL (v: [d: [nat, nat] -> above(d'1+d'2)], m, n: nat):
    n=0 AND NOT m=0 IMPLIES v(m-1, 1) > m+n;
```

```
ack_gt_m_n_TCC7: OBLIGATION
  FORALL (v: [d: [nat, nat] -> above(d'1+d'2)], m, n: nat):
    NOT n=0 AND NOT m=0 IMPLIES v(m-1, v(m, n-1)) > m+n;
```


PVS Automatically Uses Judgements

Most of these TCCs are automatically discharged by the type checker (in this case, all of them). Furthermore, the theorem prover automatically uses judgements:

```
ack_simple_property :
```

```
  |-----
{1}  FORALL (m, n): ack(m, n) > max(m, n)
```

Rule? (grind)

Rewriting with max

Trying repeated skolemization, instantiation, and if-lifting,
Q.E.D.

For Loops

```
/* Pre: n >= 0 */
int a = 1;
for (int i=0; i < n; i++) {
  /* Inv: a = i! */
  a = a*(i+1);
}
/* Post: a = n! */
```

In PVS:

```
IMPORTING structures@for_iterate
```

```
fact_for(n:nat) : real =
  for[real](0,n-1,1,LAMBDA(i:below(n),a:real):
    a*(i+1))
```

Proving Correctness of For Loops

Consider the following implementation of factorial:

```
fact_for : THEOREM
  fact_for(n) = factorial(n)
```

```
fact_for :
```

```
  |-----
{1}  FORALL (n: nat): fact_for(n) = factorial(n)
```

Rule? (skeep)(expand "fact_for")

```
fact_for :
```

```
  |-----
{1}  for[real] (0,n-1,1,LAMBDA (i:below(n),a:real):a+a*i) =
      factorial(n)
```

Rule? (lemma "for_induction[real]")

Applying for_induction[real]

this simplifies to:

fact_for :

```
{-1}  FORALL (i, j: int, a: real, f: ForBody[real](i, j),
           inv: PRED[[UpTo[real](1 + j - i), real]]):
      (inv(0, a) AND
       (FORALL (k: subrange(0, j - i), ak: real):
         inv(k, ak) IMPLIES inv(k + 1, f(i + k, ak))))
      IMPLIES inv(j - i + 1, for(i, j, a, f))
|-----
[1]   for[real](0,n-1,1,LAMBDA (i:below(n),a:real):a+a*i) =
      factorial(n)
```

Rule? (inst?)

Instantiating quantified variables,
this yields 2 subgoals:

fact_for.1 :

```
{-1}  FORALL (inv:PRED[[UpTo[real](n)real]]):
      (inv(0,1) AND
       (FORALL (k:subrange(0,n-1),ak:real):
         inv(k,ak) IMPLIES inv(k+1,ak+ak*(0+k))))
      IMPLIES
      inv(n,
         for(0,n-1,1,LAMBDA (i:below(n),a:real):a+a*i))
|-----
[1]   for[real](0,n-1,1,LAMBDA (i:below(n),a:real):a+a*i) =
      factorial(n)
```

```
Rule? (inst -1 "LAMBDA(i:upto(n),a:real) : a = factorial(i)")
fact_for.1.1 :
```

```
{-1} ...
```

```
|-----
```

```
[1]   for[real](0,n-1,1,LAMBDA (i:below(n),a:real):a+a*i) =
      factorial(n)
```

- ▶ The variable i in the invariant refers to the i th iteration.
- ▶ Remaining subgoals are discharged with (grind). See Examples/Lecture-2.pvs.

Exercise 3

Implement in PVS the following algorithm that finds the fastest aircraft in the NAS:

```
/* Pre: nas is non empty */
ac = nas[1]
for (int i=1; i < n; i++) {
  /* Inv: ac = max(ac[1..i]) */
  ac = max(ac,nas[i+1]);
}
/* Post: ac = max(nas[1..n]) */
```

Exercise 3 (cont.)

Prove the following property:

```
fastest_correct : THEOREM
  FORALL (nas:(nonempty nas?), i:subrange(1,nas'n)):
    fastest(nas)'gs >= nas'seq(i)'gs
```

Hint: You may need the following invariant:

```
LAMBDA(i:upto(nas'n-1), ac:Aircraft):
  FORALL (k:subrange(1,i+1)): ac'gs >= nas'seq(k)'gs
```


Inductive Definitions

- ▶ An inductive definition gives rules for generating members of a set.
- ▶ An object is in the set, only if it has been generated according to the rules.
- ▶ An inductively defined set is the smallest set closed under the rules.
- ▶ PVS automatically generates weak and strong induction schemes that are used by command `(rule-induct "<name>")` command .

Even and Odd

```
even(n:nat): INDUCTIVE bool =  
  n = 0 OR (n > 1 AND even(n - 2))
```

```
odd(n:nat): INDUCTIVE bool =  
  n = 1 OR (n > 1 AND odd(n - 2))
```

Induction Schemes

The definition of `even` generates the following induction schemes (use the Emacs command `M-x ppe`):

```

even_weak_induction: AXIOM
  FORALL (P: [nat -> boolean]):
    (FORALL (n: nat): n = 0 OR (n > 1 AND P(n - 2))
      IMPLIES P(n))
  IMPLIES
    (FORALL (n: nat): even(n) IMPLIES P(n));

even_induction: AXIOM
  FORALL (P: [nat -> boolean]):
    (FORALL (n: nat):
      n = 0 OR (n > 1 AND even(n - 2) AND P(n - 2))
      IMPLIES P(n))
  IMPLIES (FORALL (n: nat): even(n) IMPLIES P(n));

```

Inductive Proof

```
even_odd :
```

```
  |-----
{1}  FORALL (n: nat): even(n) => odd(n + 1)
```

Rule? (rule-induct "even")

Applying rule induction over even, this simplifies to:

```
even_odd :
```

```
  |-----
{1}  FORALL (n: nat):
      n = 0 OR (n > 1 AND odd(n - 2 + 1)) IMPLIES odd(n + 1)
```

The proof can then be completed using

```
(skosimp*)(rewrite "odd" +)(ground)
```

Mutual Recursion and Higher-Order Recursion

The predicates `odd` and `even` can be defined using a mutual-recursion:

$$\text{even?}(0) = \text{true}$$
$$\text{odd?}(0) = \text{false}$$
$$\text{odd?}(1) = \text{true}$$
$$\text{even?}(n + 1) = \text{odd?}(n)$$
$$\text{odd?}(n + 1) = \text{even?}(n)$$

In PVS ...

```
my_even?(n) : INDUCTIVE bool =  
  n = 0 OR n > 0 AND my_odd?(n-1)
```

```
my_odd?(n) : INDUCTIVE bool =  
  n = 1 OR n > 1 AND my_even?(n-1)
```

- ▶ These definitions don't type-check. What is wrong with them?
- ▶ **PVS does not (directly) support mutual recursion.**

Mutual Recursion via Higher-Order Recursion

```
even_f?(fodd: [nat->bool],n) : bool =  
  n = 0 OR  
  n > 0 AND fodd(n-1)
```

```
my_odd?(n) : INDUCTIVE bool =  
  n = 1 OR  
  n > 1 AND even_f?(my_odd?,n-1)
```

```
my_even?(n) : bool =  
  even_f?(my_odd?,n)
```

The only recursive definition is my_odd?

Exercise 4

Prove the following equivalences:

```
my_even_my_odd : LEMMA
  my_even?(n) = even(n) AND
  my_odd?(n) = odd(n)
```

Hint: Use induction on n and lemmas `even_odd` and `odd_even` (see `Exercises/Lecture-2.pvs`).

Inductive Abstract Data Types

PVS supports ADTs (**constructors**, **recognizers**, **selectors**):

```
Tree : DATATYPE
BEGIN
  nulltree : nulltree?
  constree(val:int,left:Tree,right:Tree) : constree?
END Tree
```

Inductive Abstract Data Types

For each datatype declaration, PVS automatically generates a well-founded structural order \ll , an induction scheme, and ... (see the automatically generated file `Tree_adt.pvs`)

```
Tree_induction: AXIOM
  FORALL (p: [Tree -> boolean]):
    (p(nulltree) AND
      (FORALL (constree1_var: int, constree2_var: Tree,
              constree3_var: Tree):
        p(constree2_var) AND p(constree3_var) IMPLIES
          p(constree(constree1_var, constree2_var,
                    constree3_var))))
    IMPLIES (FORALL (Tree_var: Tree): p(Tree_var));
```

Pattern Matching Support

```
height(t:Tree) : RECURSIVE nat =  
  CASES t OF  
    constree(v,l,r): 1+max(height(l),height(r))  
    ELSE 0  
  ENDCASES  
MEASURE t BY <<
```

ADT are First Class Terms

```
monotonetree?(t:Tree): INDUCTIVE bool =  
  nulltree?(t) OR  
  (constree?(left(t)) IMPLIES val(t) > val(left(t))) AND  
  (constree?(right(t)) IMPLIES val(t) > val(right(t))) AND  
  monotonetree?(left(t)) AND  
  monotonetree?(right(t))
```

```
consotonetree?(t:Tree): MACRO bool =  
  constree?(t) AND monotonetree?(t)
```

Exercise 5

Prove the following property (see Exercises/Lecture-2.pvs):

```
height_monotone: THEOREM
  FORALL (t:(consotonetree?)):
    height(t) <= val(t)+1
```