A PVS Graph Theory Library

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Abstract

This paper documents the NASA Langley PVS graph theory library. The library provides fundamental definitions for graphs, subgraphs, walks, paths, subgraphs generated by walks, trees, cycles, degree, separating sets, and four notions of connectedness. Theorems provided include Ramsey’s and Menger’s and the equivalence of all four notions of connectedness.

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1 Introduction

This paper documents the NASA Langley PVS graph theory library. The library develops the fundamental concepts and properties of finite graphs.

2 Definition of a Graph

The standard mathematical definition of a graph is that it is an ordered pair of sets $(V,E)$ such that $E$ is a subset of the ordered pairs of $V$. Typically $V$ and $E$ are assumed to be finite though sometimes infinite graphs are treated as well. The NASA library is restricted to finite graphs only. The set $V$ is called the vertices of the graph and the set $E$ is called the edges of the graph.

Although PVS directly supports ordered pairs, we have chosen the PVS record structure to define a graph. The advantage of the record structure is that it provides names for the vertex and edge sets rather than $\text{proj} \_1$ and $\text{proj} \_2$. For efficiency reasons, it is preferable to define a graph in PVS in two steps. We begin with the definition of a pregraph:

$$\text{pregraph: TYPE} = \[\# \ \text{vert: finite_set[T]}, \ \text{edges: finite_set[doubleton[T]]}\ #\]$$

A pregraph is a structured type with two components: vert and edges. The vert component is a finite set over an arbitrary type $T$. This represents the vertices of the graph. The edges component is a finite set of doubletons (i.e. sets with exactly two members) of $T$. Thus, an edge is defined by designating its two end vertices. The type $\text{finite_set}$ is defined in the PVS finite sets library. It is a subtype of the type $\text{set}$ which is defined in the PVS prelude as follows:

$$\text{sets [T: TYPE]: THEORY}$$

$$\text{BEGIN}$$

$$\text{set: TYPE} = \[T \to \text{bool}\]$$

$$x, y: \text{VAR T}$$

$$a, b, c: \text{VAR set}$$

$$p: \text{VAR [T \to \text{bool}]}$$

$$\text{member}(x, a): \text{bool} = a(x)$$

$$\text{emptyset: set} = x \mid \text{false}$$

$$\text{subset?}(a, b): \text{bool} = (\text{FORALL} \ x: \text{member}(x, a) \Rightarrow \text{member}(x, b))$$

$$\text{union}(a, b): \text{set} = x \mid \text{member}(x, a) \ OR \ \text{member}(x, b)$$

$$\text{intersection}(a, b): \text{set} = x \mid \text{member}(x, a) \ AND \ \text{member}(x, b)$$

$$\text{END sets}$$

A set is just a boolean-valued function of the element type, i.e., a function from $T$ into bool. In PVS this is written as $[T \to \text{bool}]$. If $x$ is a member of a set $S$, the expression $S(x)$ evaluates to true, otherwise it evaluates to false.

Finite sets are defined as follows:
$S$: VAR set[T]

is_finite($S$): bool = (EXISTS (N: nat, f: [(S) -> below[N]]): injective?(f))

finite_set: TYPE = \{ S | is_finite(S) \} CONTAINING emptyset[T]

Thus finite sets are sets which can be mapped onto $0...N$ for some $N$. The cardinality function card is defined as follows:

inj_set($S$): (nonempty?[nat]) =
\{ n | EXISTS (f : [(S)->below[n]]): injective?(f) \}

card($S$): nat = min(inj_set($S$))

All of the standard properties about card have been proved and are available:

card_union : THEOREM card(union(A,B)) = card(A) + card(B) -
card(intersection(A,B))

card_add : THEOREM card(add(x,S)) = card(S) +
IF S(x) THEN 0 ELSE 1 ENDIF

card_remove : THEOREM card(remove(x,S)) = card(S) -
IF S(x) THEN 1 ELSE 0 ENDIF

card_subset : THEOREM subset?(A,B) IMPLIES card(A) <= card(B)

card_emptyset : THEOREM card(emptyset[T]) = 0

card_singleton: THEOREM card(singleton(x)) = 1

Now we are ready to define a graph as follows:

graph: TYPE = \{ g: pregraph | (FORALL (e: doubleton[T]): edges(g)(e) IMPLIES
(FORALL (x: T): e(x) IMPLIES vert(g)(x)) \}

A graph is a pregraph where the edges set contains doubleton sets with elements restricted to the vert set. The doubleton type is defined as follows:

doubletons[T: TYPE]: THEORY
BEGIN

x,y,z: VAR T

dbl(x,y): set[T] = \{ t: T | t = x OR t = y \}
S: VAR set[T]

doubleton?(S): bool = (EXISTS x,y: x /= y AND S = dbl(x,y))

doubleton: TYPE = {S | EXISTS x,y: x /= y AND S = dbl(x,y)}
END doubletons

For example, suppose the base type T is defined as follows:

T: TYPE = {a,b,c,d,e,f,g}

Then the following pregraph is also a graph:

(# vert := {a,b,c},
 edges := { {a,b}, {b,c} } #)

whereas

(# vert := {a,b,c},
 edges := { {a,b}, {b,d}, {a,g} } #)

is a pregraph but is not a graph

1

The size of a graph is defined as follows:

size(G): nat = card[T](vert(G))

A singleton graph with one vertex x (i.e. size is 1) can be constructed using the following function:

singleton_graph(v): graph = (# vert := singleton[T](v),
 edges := emptyset[doubleton[T]] #)

For convenience we define a number of predicates:

edge?(G)(x,y): bool = x /= y AND edges(G)(dbl[T](x,y))

empty?(G): bool = empty?(vert(G))

singleton?(G): bool = (size(G) = 1)

isolated?(G): bool = empty?(edges(G))

The net result is that we have the following:

1PVS does not allow { .. } as set constructors. These must be constructed in PVS using LAMBDA expressions or through use of the functions add, emptyset, etc.
vert(G) vertices of graph G (a finite set of T)
edges(G) edges of a graph G (a finite set of doubletons taken from vert(G))
edge?(G)(x,y) true IFF there is an edge between vertices x and y
empty?(G) true IFF the graph G has no vertices
singleton?(G) true IFF graph G has only 1 vertex
isolated?(G) true IFF graph G has no edges

The following useful lemmas are provided:

\[ x, y, v : \text{VAR } T \]
\[ e : \text{VAR doubleton}[T] \]
G: var graph

edge?_comm : LEMMA edge?(G)(y, x) IMPLIES edge?(G)(x, y)
edge_has_2_verts : LEMMA x /= v AND e(x) AND e(v) IMPLIES e = dbl(x,v)
edge_in_card_gt_1 : LEMMA edges(G)(e) IMPLIES card(vert(G)) > 1
not_singleton_2_vert : LEMMA NOT empty?(G) AND NOT singleton?(G)
IMPLIES (EXISTS (x,y: T): x /= y AND vert(G)(x) AND vert(G)(y))

These definitions and lemmas are located in the graphs theory.

3 Graph Operations

The theory graph_ops defines the following operations on a graph:

union(G1,G2) creates a graph that is a union of G1 and G2
del_vert(G,v) removes vertex v and all adjacent edges to v from the graph G
del_edge(e,G) creates subgraph with edge e removed from G

These operations are defined as follows:

\[ \text{union}(G1,G2) : \text{graph}[T] = (# \text{ vert := union(vert(G1),vert(G2))}, \]
\[ \text{edges := union(edges(G1),edges(G2)) } \) \]

\[ \text{del_vert}(G: \text{graph}[T], v: T) : \text{graph}[T] = \]
\[ (# \text{ vert := remove[T](v,vert(G))}, \]
\[ \text{edges := e | edges(G)(e) AND NOT e(v) } \) \]

\[ \text{del_edge}(e,G) : \text{graph}[T] = G \text{ WITH } [\text{edges := remove(e,edges(G))} ] \]

The following is a partial list of the properties that have been proved:
del.vert.still_in : LEMMA FORALL (x: (vert(G))):
    x /= v IMPLIES vert(del.vert(G, v))(x)

size_del_vert : LEMMA FORALL (v: (vert(G))):
    size(del.vert(G, v)) = size(G) - 1

del.edge.lem3 : LEMMA edges(G)(e) AND e2 /= e IMPLIES
    edges(del.edge(G, e))(e2)

4 Graph Degree

The theory graph_deg develops the concept of degree of a vertex. The following functions are defined:

incident_edges(v, G) returns set of edges attached to vertex v in graph G
deg(v, G) number of edges attached to vertex v in graph G

Formally they are specified as follows:

v: VAR T
G,GS: VAR graph[T]

incident_edges(v, G) : finite_set[doubleton[T]] = {e: doubleton[T] | edges(G)(e) AND e(v) }

deg(v, G): nat = card(incident_edges(v, G))

The following useful properties are proved

deg.del.edge : LEMMA e = dbl(x,y) AND edges(G)(e) IMPLIES
    deg(y, G) = deg(y, del.edge(G, e)) + 1

deg.edge.exists : LEMMA deg(v, G) > 0 IMPLIES
    (EXISTS e: e(v) AND edges(G)(e))
deg_to_card : LEMMA deg(v, G) > 0 IMPLIES size(G) >= 2

del_vert_deg_0 : LEMMA deg(v, G) = 0 IMPLIES edges(del_vert(G, v)) = edges(G)

deg_del_vert : LEMMA x /= v AND edges(G)(dbl[T](x, v))
    IMPLIES deg(v, del_vert(G, x)) =
        deg(v, G) - 1

del_vert_not_incident: LEMMA x /= v AND NOT edges(G)(dbl[T](x, v)) IMPLIES
    deg(x, del_vert(G, v)) = deg(x, G)

singleton_deg: LEMMA singleton?(G) IMPLIES deg(v, G) = 0

5 Subgraphs

The subgraph relation is defined as a predicate named subgraph?:

G1, G2: VAR graph[T]

subgraph?(G1, G2): bool = subset?(vert(G1), vert(G2)) AND
    subset?(edges(G1), edges(G2))

The subgraph type is defined using this predicate:

Subgraph(G: graph[T]): TYPE = { S: graph[T] | subgraph?(S, G) }

The subgraph generated by a vertex set is defined as follows:

i: VAR T
e: VAR doubleton[T]
subgraph(G, V): Subgraph(G) =
    (G WITH [vert := {i | vert(G)(i) AND V(i)},
        edges := {e | edges(G)(e) AND
            (FORALL (x: T): e(x) IMPLIES V(x)) }])

The following properties have been proved:

finite_vert_subset : LEMMA is_finite(LAMBDA (i: T): vert(G)(i) AND V(i))

subgraph_vert_sub : LEMMA subset?(V, vert(G)) IMPLIES
    vert(subgraph(G, V)) = V
subgraph_lem : LEMMA subgraph?(subgraph(G,V),G)

SS: VAR graph[T]
subgraph_smaller : LEMMA subgraph?(SS, G) IMPLIES size(SS) <= size(G)

These definitions and lemmas are located in the subgraphs theory.

6 Walks and Paths

Walks are defined using finite sequences which are defined in the seq_def theory:

seq_def[T: TYPE]: THEORY
BEGIN
finite_seq: TYPE = [# l: nat, seq: [below[l] -> T] #]
END

We begin by defining a prewalk as follows:

prewalk: TYPE = {w: finite_seq[T] | l(w) > 0}

where, as before, T is the base type of vertices. A prewalk is a finite sequence of vertices.
Thus, if we make the declaration:

w: VAR prewalk

l(w) is the length of the prewalk and seq(w)(i) is the ith element in the sequence. Prewalks
are constrained to be greater than 1 in length. We have used the PVS conversion mechanism,
so that w(i) can be written instead of seq(w)(i). A walk is then defined as follows:

s,ps,ww: VAR prewalk

verts_in?(G,s): bool = (FORALL (i: below(l(s))): vert(G)(seq(s)(i)))

walk?(G,ps): bool = verts_in?(G,ps) AND (FORALL n: n < l(ps) - 1 IMPLIES edge?(G)(ps(n),ps(n+1)))

Seq(G) : TYPE = {w: prewalk | verts_in?(G,w)}
Walk(G): TYPE = {w: prewalk | walk?(G,w)}
A walk is just a prewalk where all of the vertices are in the graph and there is an edge between each consecutive element of the sequence. The dependent type $\text{Walk}(G)$ defines the domain (or type) of all walks in a graph $G$. The dependent type $\text{Seq}(G)$ defines the domain (or type) of all prewalks in a particular graph $G$.

The predicates $\text{from?}$ and $\text{walk_from?}$ identify sequences and walks from one particular vertex to another.

$$\text{from?(ps,u,v)}: \text{bool} = \text{seq(ps)}(0) = u \text{ AND seq(ps)}(1(ps) - 1) = v$$

$$\text{walk_from?}(G,ps,u,v): \text{bool} =$$

$$\text{seq(ps)}(0) = u \text{ AND seq(ps)}(1(ps) - 1) = v \text{ AND walk?(G,ps)}$$

The function $\text{verts_of}$ returns the set of vertices that are in a walk:

$$\text{verts_of}(ww: \text{prewalk}): \text{finite_set}[T] =$$

$$\{t: T \mid (\text{EXISTS (i: below(l(ww))): } \text{ww}(i) = t)\}$$

Similarly, the function $\text{edges_of}$ returns the set of edges that are in a walk:

$$\text{edges_of}(ww): \text{finite_set}[\text{doubleton}[T]] = \{e: \text{doubleton}[T] \mid$$

$$\text{EXISTS (i: below(l(ww)-1)): } e = \text{dbl}(\text{ww}(i),\text{ww}(i+1))\}$$

Below are listed some of the proved properties about walks:

$$G,GG: \text{VAR}\ G\text{raph}[T]$$

$$x,u,v: \text{VAR}\ T$$

$$i,j,n: \text{VAR}\ \text{nat}$$

$$ps: \text{VAR}\ \text{prewalk}$$

$$\text{verts_in_concat}: \text{LEMMA}\ \text{FORALL}\ (s1,s2: \text{Seq}(G)): \text{verts_in?(G,s1 o s2)}$$

$$\text{verts_in_caret}: \text{LEMMA}\ \text{FORALL}\ (j: \text{below}(l(ps))): i <= j \text{ IMPLIES verts_in?(G,ps) IMPLIES verts_in?(G,ps^i)(i,j))}$$

$$\text{vert_seq_lem}: \text{LEMMA}\ \text{FORALL}\ (w: \text{Seq}(G)): n < l(w) \text{ IMPLIES vert(G)(w(n))}$$

$$\text{verts_of_subset}: \text{LEMMA}\ \text{FORALL}\ (w: \text{Seq}(G)): \text{subset?(verts_of}(w),\text{vert}(G))$$

$$\text{edges_of_subset}: \text{LEMMA}\ \text{walk?(G,ps) IMPLIES subset?(edges_of(ps),edges}(G))$$

$$\text{walk_verts_in}: \text{LEMMA}\ \text{walk?(G,ps) IMPLIES verts_in?(G,ps)}$$

$$\text{walk_from_vert}: \text{LEMMA}\ \text{FORALL}\ (w: \text{prewalk},v1,v2:T):$$
walk_from?(G,w,v1,v2) IMPLIES
   vert(G)(v1) AND vert(G)(v2)

walk_edge_in : LEMMA walk?(G,ps) AND
   subset?(edges_of(ps),edges(GG)) AND
   subset?(verts_of(ps),vert(GG))
IMPLIES walk?(GG,ps)

The walks theory also proves some useful operators for walks:

- gen_seq1(G, u)
  create a prewalk of length 1 consisting of a single vertex u
- gen_seq2(G, u, v)
  create a prewalk of length 2 from u to v
- trunc1(p)
  return a prewalk equal to p except the last vertex has been removed
- add1(p, v)
  return a prewalk equal to p except the vertex v has been added
- rev(p)
  return a finite sequence that is the reverse of p
- o
  concatenates two finite sequences
- ^(m,n)
  returns a finite sequence from the m .. n elements of a sequence. For example if p = v0 -> v1 -> v2 -> v3 -> v4, then p^(1,2) = v1 -> v2.

These are defined formally as follows:

\[
\text{gen_seq1}(G, (u: (vert(G)))) = \text{Seq}(G) = \\
(\# 1 := 1, \text{seq} := (\text{LAMBDA} (i: below(1)): u) ) #)
\]

\[
\text{gen_seq2}(G, (u,v: (vert(G)))) = \text{Seq}(G) = \\
(\# 1 := 2, \\
  \text{seq} := (\text{LAMBDA} (i: below(2)): \\
   \text{IF} i = 0 \text{ THEN} u \text{ ELSE} v \text{ ENDIF} ) ) #)
\]

\[
\text{Longprewalk: TYPE} = \{ps: \text{prewalk} \mid l(ps) \geq 2\}
\]

\[
\text{trunc1}(p: \text{Longprewalk}) = \text{prewalk} = p^-(0, l(p)-2)
\]

\[
\text{add1}(ww,x): \text{prewalk} = (# 1 := l(ww) + 1, \\
  \text{seq} := (\text{LAMBDA} (ii: below(l(ww) + 1)): \\
   \text{IF} ii < l(ww) \text{ THEN} \text{seq}(ww)(ii) \text{ ELSE} x \text{ ENDIF} ) #)
\]

fs, fs1, fs2, fs3: VAR finite_seq
m, n: VAR nat

\[
\text{o}(fs1, fs2): \text{finite_seq} = \\
\text{LET} l1 = l(fs1), \\
  lsum = l1 + l(fs2) \\
\text{IN} (# 1 := lsum, \\
  \text{seq} := (\text{LAMBDA} (n:below[lsum])):
\]

10
IF n < l1
    THEN seq(fs1)(n)
    ELSE seq(fs2)(n-l1)
ENDIF);

emptyarr(x: below[0]): T
emptyseq: fin_seq(0) = (# 1 := 0, seq := emptyarr #);

p: VAR [nat, nat] ;

^(fs: finite_seq, (p: [nat, below(l(fs))]):
    fin_seq(IF proj_1(p) > proj_2(p) THEN 0
    ELSE proj_2(p)-proj_1(p)+1 ENDIF) =
    LET (m, n) = p
    IN IF m > n
    THEN emptyseq
    ELSE (# 1 := n-m+1,
        seq := (LAMBDA (x: below[n-m+1]): seq(fs)(x + m)) #)
    ENDIF ;

rev(fs): finite_seq = (# 1 := l(fs),
    seq := (LAMBDA (i: below(l(fs))): seq(fs)(l(fs)-l-i))
    #)

The following is a partial list of the proven properties about walks:

gen_seq1_is_walk: LEMMA vert(G)(x) IMPLIES walk?(G,gen_seq1(G,x))

dge_to_walk : LEMMA u /= v AND edges(G)(edges(T)(u, v)) IMPLIES
    walk?(G,gen_seq2(G,u,v))

dwalk?_add1 : LEMMA walk?(G,ww) AND vert(G)(x)
    AND edge?(G)(seq(ww)(l(ww)-1),x)
    IMPLIES walk?(G,add1(ww,x))

dwalk?_rev : LEMMA walk?(G,ps) IMPLIES walk?(G,rev(ps))

dwalk?_caret : LEMMA i <= j AND j < l(ps) AND walk?(G,ps)
    IMPLIES walk?(G,ps^(i,j))

yt: VAR T
p1,p2: VAR prewalk
A path is a walk that does not encounter the same vertex more than once. The predicate path\? identifies paths:

\[
p:\text{VAR prewalk}
\]

\[
\text{path}(G,ps): \text{bool} = \text{walk}(G,ps) \text{ AND } (\text{FORALL } (i,j:\text{below}(l(ps))): i \neq j \implies ps(i) \neq ps(j))
\]

Similarly the predicate path_from\? identifies paths from vertex s to t:

\[
\text{path_from}(G,ps,s,t): \text{bool} = \text{path}(G,ps) \text{ AND } \text{from}(ps,s,t)
\]

Corresponding dependent types are defined:

\[
\text{Path}(G): \text{TYPE} = \{p:\text{prewalk} \mid \text{path}(G,p)\}
\]

\[
\text{Path_from}(G,s,t): \text{TYPE} = \{p:\text{prewalk} \mid \text{path_from}(G,p,s,t)\}
\]

The following is a partial list of proven properties:

\[
G: \text{VAR graph}[T]
\]

\[
x,y,s,t: \text{VAR } T
\]

\[
i,j: \text{VAR } \text{nat}
\]

\[
p,ps: \text{VAR prewalk}
\]

\[
\text{path}_\uparrow: \text{LEMMA } i \leq j \text{ AND } j < l(ps) \text{ AND } \text{path}(G,ps) \implies \text{path}(G,ps^{(i,j)})
\]

\[
\text{path_from}_\uparrow: \text{LEMMA } i \leq j \text{ AND } j < l(ps) \text{ AND } \text{path_from}(G,ps,s,t) \implies \text{path_from}(G,ps^{(i,j)},seq(ps)(i),seq(ps)(j))
\]

\[
\text{path}_\downarrow: \text{LEMMA } \text{path}(G,ps) \implies \text{path}(G,\text{rev}(ps))
\]

\[
\text{path}_\text{gen_seq2}: \text{LEMMA } \text{vert}(G)(x) \text{ AND } \text{vert}(G)(y) \text{ AND } \text{edge}(G)(x,y) \implies \text{path}(G,\text{gen_seq2}(G,x,y))
\]

\[
\text{path}_\text{add1}: \text{LEMMA } \text{path}(G,p) \text{ AND } \text{vert}(G)(x) \text{ AND } \text{edge}(G)(seq(p)(l(p)-1),x)
\]
AND NOT verts_of(p)(x)
IMPLIES path?(G,add1(p,x))

path_trunc1 : LEMMA path?(G,p) AND l(p) > 1 IMPLIES
path_from?(G,trunc1(p),seq(p)(0),seq(p)(l(p)-2))

These definitions and lemmas about paths are located in the paths theory.

7 Connected Graphs

The library provides four different definitions for connectedness of a graph and provides proofs that they are all equivalent. These are named connected, path_connected, piece_connected, and completed:

G,G1,G2,H1,H2: VAR graph[T]

connected?(G): RECURSIVE bool = singleton?(G) OR
(EXISTS (v: (vert(G))): deg(v,G) > 0
AND connected?(del_vert(G,v)))
MEASURE size(G)

path_connected?(G): bool = NOT empty?(G) AND
(FORALL (x,y: (vert(G))):
(EXISTS (w: Walk(G)): seq(w)(0) = x AND
seq(w)(l(w)-1) = y))

piece_connected?(G): bool = NOT empty?(G) AND
(FORALL H1,H2: G = union(H1,H2) AND
NOT empty?(H1) AND NOT empty?(H2)
IMPLIES NOT empty?(intersection(vert(H1),
vert(H2))))

completed?(G): bool = IF isolated?(G) THEN singleton?(G)
ELSIF (EXISTS (v: (vert(G))): deg(v,G) = 1) THEN
(EXISTS (x: (vert(G))): deg(x,G) = 1 AND
connected?(del_vert(G,x)))
ELSE
(EXISTS (e: (edges(G))):
connected?(del_edge(G,e)))
ENDIF

These definitions are located in the graph_conn_defs theory. The following lemmas about equivalence are located in the theory graph_connected:

graph_connected[T: TYPE]: THEORY
8 Circuits

A slightly non-traditional definition of circuit is used. A circuit is a walk that starts and ends in the same place (i.e. a pre_circuit) and is cyclically reduced (i.e. cyclically_reduced?).

reducible?(G: graph[T], w: Seq(G)): bool = (EXISTS (k: posnat): k < l(w) - 1 AND w(k-1) = w(k+1))

reduced?(G: graph[T], w: Seq(G)): bool = NOT reducible?(G,w)

cyclically_reduced?(G: graph[T], w: Seq(G)): bool = l(w) > 2 AND reduced?(G,w) AND w(1) /= w(l(w)-2)

pre_circuit?(G: graph[T], w: prewalk): bool = walk?(G,w) AND w(0) = w(l(w)-1)

circuit?(G: graph[T], w: Seq(G)): bool = walk?(G,w) AND cyclically_reduced?(G,w) AND pre_circuit?(G,w)

The following properties are proved in the circuit_deg theory:

cir_deg_G : LEMMA (EXISTS (a,b: (vert(G))): vert(G)(z) AND a /= b AND edge?(G)(a,z) AND edge?(G)(b,z) ) IMPLIES deg(z,G) >= 2

circuit_deg : LEMMA FORALL (w: Walk(G),i: below(l(w))): circuit?(G,w) IMPLIES deg(w(i),G_from(G,w)) >= 2
9 Trees

Trees are defined recursively as follows:

\[ G: \text{VAR} \ \text{graph}[T] \]

\[ \text{tree?}(G) : \text{RECURSIVE} \ \text{bool} = \text{card}([T](\text{vert}(G))) = 1 \text{ OR } \]
\[ \quad \text{EXISTS} \ (v: (\text{vert}(G))): \text{deg}(v, G) = 1 \text{ AND } \]
\[ \quad \text{tree?}(\text{del_vert}[T](G, v)) \]

\[ \text{MEASURE} \ \text{size}(G) \]

and the Tree type is defined as follows:

\[ \text{Tree} : \text{TYPE} = \{G: \text{graph}[T] \mid \text{tree?}(G)\} \]

The fundamental property that trees have no circuits is proved in \text{tree_circ} theory.

\[ \text{tree_no_circuits} : \text{THEOREM} \ (\text{FORALL} \ (w: \text{Walk}(G)): \text{tree?}(G) \Rightarrow \]
\[ \quad \neg \text{circuit?}(G, w)) \]

10 Ramsey’s Theorem

This work builds upon a verification of this theorem by Natarajan Shankar and the paper entitled “The Boyer-Moore Prover and Nuprl: An Experimental Comparison” by David Basin and Matt Kaufmann2.

\[ i, j: \text{VAR} \ T \]
\[ n, p, q, ii: \text{VAR} \ \text{nat} \]
\[ g: \text{VAR} \ \text{graph}[T] \]
\[ G: \text{VAR} \ \text{Graph}[T] \ % \text{nonempty} \]
\[ V: \text{VAR} \ \text{finite_set}[T] \]

\[ \text{contains_clique}(g, n) : \text{bool} = \]
\[ \quad \text{EXISTS} \ (C: \text{finite_set}[T]): \]
\[ \quad \text{subset?}(C, \text{vert}(g)) \text{ AND card}(C) \geq n \text{ AND } \]
\[ \quad \text{FORALL} \ i, j: i \neq j \text{ AND } C(i) \text{ AND } C(j) \text{ IMPLIES } \text{edge?}(g)(i, j)) \]

\[ \text{contains_indep}(g, n) : \text{bool} = \]
\[ \quad \text{EXISTS} \ (D: \text{finite_set}[T]): \]

subset?(D, vert(g)) AND card(D) >= n AND
(FORALL i, j: i/=j AND D(i) AND D(j) IMPLIES NOT edge?(g)(i, j))

subgraph_clique: LEMMA (FORALL (V: set[T]):
contains_clique(subgraph(g, V), p)
IMPLIES contains_clique(g, p))

subgraph_indep : LEMMA (FORALL (V: set[T]):
contains_indep(subgraph(g, V), p)
IMPLIES contains_indep(g, p))

ramseys_theorem: THEOREM (EXISTS (n: posnat):
(FORALL (G: Graph[T]): size(G) >= n
IMPLIES (contains_clique(G, 11) OR
contains_indep(G, 12))))

11 Menger’s Theorem

To state menger’s theorem one must first define minimum separating sets. This is fairly complicated in a formal system. We begin with the concept of a separating set:

G: VAR graph[T]
v,s,t: VAR T
e: VAR doubleton[T]
V: VAR finite_set[T]

del_verts(G,V): graph[T] =
(# vert := difference[T](vert(G),V),
edges := {e | edges(G)(e) AND
(FORALL v: V(v) IMPLIES NOT e(v))} #)

separates(G,V,s,t): bool = NOT V(s) AND NOT V(t) AND
NOT (EXISTS (w: prewalk): walk_from?(del_verts(G,V),w,s,t))

In other words V separates s and t when its removal disconnects s and t. To define the minimum separating set, we use an abstract minimum function defined in the abstract_min theory. The net result is that we end up with a function min_sep_set with all of the following desired properties

min_sep_set(G,s,t): finite_set[T] = min[seps(G,s,t),
(LAMBDA (v: seps(G,s,t)): card(v)),
(LAMBDA (v: seps(G,s,t)): true)]
separable?(G,s,t): bool = (s /= t AND NOT edge?(G)(s,t))

min_sep_set_edge: LEMMA NOT separable?(G,s,t) IMPLIES
               min_sep_set(G,s,t) = vert(G)

min_sep_set_card: LEMMA FORALL (s,t: (vert(G))): separates(G,V,s,t)
                 IMPLIES card(min_sep_set(G,s,t)) <= card(V)

min_sep_set_seps: LEMMA separable?(G,s,t) IMPLIES
                 separates(G,min_sep_set(G,s,t),s,t)

min_sep_set_vert: LEMMA separable?(G,s,t) AND min_sep_set(G,s,t)(v)
                 IMPLIES vert(G)(v)

ends_not_in_min_sep_set: LEMMA separable?(G,s,t) AND min_sep_set(G, s, t)(v)
                          IMPLIES v /= s AND v /= t

We then define sep_num as follows:

   sep_num(G,s,t): nat = card(min_sep_set(G,s,t))

Next, we define a predicate independent? that defines when two paths are independent:

   independent?(w1,w2: prewalk): bool =
       (FORALL (i,j: nat): i > 0 AND i < l(w1) - 1 AND
        j > 0 AND j < l(w2) - 1 IMPLIES
        seq(w1)(i) /= seq(w2)(j))

The concept of a set of independent paths is defined as follows:

   set_of_paths(G,s,t): TYPE = finite_set[Path_from(G,s,t)]

   ind_path_set?(G,s,t,(pset: set_of_paths(G,s,t))): bool =
      (FORALL (p1,p2: Path_from(G,s,t)):
       pset(p1) AND pset(p2) AND p1 /= p2
       IMPLIES independent?(p1,p2))

In other words, a set of paths is an ind_path_set? if all pairs of paths in the set are independent. We can now state Menger’s theorem in both directions:

   easy_menger: LEMMA FORALL (ips: set_of_paths(G,s,t)):
    separable?(G,s,t) AND
    ind_path_set?(G,s,t,ips) IMPLIES
    card(ips) <= sep_num(G,s,t)
hard_menger: Axiom \(\text{separable}(G, s, t) \land \text{sep_num}(G, s, t) = K \land \text{vert}(G)(s) \land \text{vert}(G)(t)\) implies 
\[(\exists \text{ips: set_of_paths}(G, s, t)) : \text{card}(\text{ips}) = K \land \text{ind_path_set}(G, s, t, \text{ips})]\]

The hard direction of Menger has only been formally proved for the \(K = 2\) case.

hard_menger: Lemma \(\text{separable}(G, s, t) \land \text{sep_num}(G, s, t) = 2 \land \text{vert}(G)(s) \land \text{vert}(G)(t)\) implies 
\[(\exists \text{ips: set_of_paths}(G, s, t)) : \text{card}(\text{ips}) = 2 \land \text{ind_path_set}(G, s, t, \text{ips})]\]

12 PVS Theories

The following is a list of the PVS theories and description:
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The PVS specifications are available at:

13 Concluding Remarks

This paper gives a brief overview of the NASA Langley PVS Graph Theory Library. The library provides definitions and lemmas for graph operations such as deleting a vertex or edge, provides definitions for vertex degree, subgraphs, minimal subgraphs, walks and paths, notions of connectedness, circuit and trees. Both Ramsey’s Theorem and Menger’s Theorem are provided.

A APPENDIX: Other Supporting Theories

A.1 Graph Inductions

The graph theory library provides two basic means of performing induction on a graph: induction on the number of vertices and induction on the number of edges.

\[ G, GG: \text{VAR graph}\{T\} \]
\[ P: \text{VAR pred[graph}\{T\}] \]

\[
\text{graph_induction_vert} : \text{THEOREM (FORALL G:}
\]
\[
\quad (\text{FORALL GG: size(GG) < size(G) IMPLIES P(GG))}
\]
\[
\quad \text{IMPLIES P(G))}
\]
\[
\quad \text{IMPLIES (FORALL G: P(G))}
\]

\[
\text{graph_induction_edge} : \text{THEOREM (FORALL G:}
\]
\[
\quad (\text{FORALL GG: num_edges(GG) < num_edges(G) IMPLIES P(GG))}
\]
\[
\quad \text{IMPLIES P(G))}
\]
\[
\quad \text{IMPLIES (FORALL G: P(G))}
\]

These theorems can be invoked using the PVS strategy \text{INDUCT}. For example

\[
\text{(INDUCT "G" 1 "graph_induction_vert")}
\]

invokes vertex induction on formula 1. They are available in theory \text{graph_inductions}. These induction theorems were proved by rewriting with the following lemmas

\[
\text{size_prep} : \text{LEMMA (FORALL G : P(G)) IFF}
\]
\[
\quad (\text{FORALL n, G : size(G) = n IMPLIES P(G))}
\]

\[
\text{num_edges_prep} : \text{LEMMA (FORALL G : P(G)) IFF}
\]
\[
\quad (\text{FORALL n, G : num_edges(G) = n IMPLIES P(G))}
\]

which converts the theorem into formulas that are universally quantified over the naturals. The resulting formulas were then easily proved using PVS’s built-in theorem for strong induction:
A.2 Subgraphs Generated From Walks

The graph theory library provides a function $G_{\text{from}}$ that constructs a subgraph of a graph $G$ that contains the vertices and edges of a walk $w$: 

$$ G_{\text{from}}(G, \text{graph}[T], w: \text{Walk}(G)): \text{Subgraph}(G) = (\# \ \text{vert} \ := \ \text{verts}_{\text{of}}(w), \newline \quad \text{edges} := \ \text{edges}_{\text{of}}(w) \ #) $$

The following properties of $G_{\text{from}}$ have been proved:

- $\text{vert}_{G_{\text{from}}}$ : LEMMA FORALL (w: Walk(G), i: below(l(w))): \newline \quad vert(G_{\text{from}}(G, w))(w(i))
- $\text{edge?}_{G_{\text{from}}}$ : LEMMA FORALL (w: Walk(G), i: below(l(w)-1)): \newline \quad edge?(G_{\text{from}}(G, w))(w(i), w(i+1))
- $\text{vert}_{G_{\text{from}}\_not}$ : LEMMA FORALL (w: Walk(G)): \newline \quad \text{subset}?(\text{vert}(G_{\text{from}}(G, w)), \text{vert}(GG)) \ \text{AND} \newline \quad \text{NOT} \ \text{verts}_{\text{of}}(w)(v) \ \text{IMPLIES} \newline \quad \text{subset}?(\text{vert}(G_{\text{from}}(G, w)), \text{remove}[T](v, \text{vert}(GG)))
- $\text{del}_{\text{vert}}_{\text{subgraph}}$: LEMMA FORALL (w: Walk(G), v: (\text{vert}(GG))): \newline \quad \text{subgraph?}(G_{\text{from}}(G, w), GG) \ \text{AND} \newline \quad \text{NOT} \ \text{verts}_{\text{of}}(w)(v) \ \text{IMPLIES} \newline \quad \text{subgraph?}(G_{\text{from}}(G, w), \text{del}_{\text{vert}}(GG, v))$

This lemmas are available in the theory $\text{subgraphs}_\text{from}_\text{walk}$.

A.3 Maximum Subgraphs

Given a graph $G$ we say that a subgraph $S$ is maximal with respect to a particular property $P$ if it is the largest subgraph that satisfies the property. Formally we write:

$$ \text{maximal?}(G: \text{graph}[T], S: \text{Subgraph}(G), P: \text{Gpred}(G)): \text{bool} = P(S) \ \text{AND} \newline (\text{FORALL} (SS: \text{Subgraph}(G)): P(SS) \ \text{IMPLIES} \newline \quad \text{size}(SS) \leq \text{size}(S)) $$
We can define a function that returns the maximum subgraph under the assumption that there exists at least one subgraph that satisfies the predicate. Therefore this function is only defined on a subtype of \( P \), namely \( G_{\text{pred}} \):

\[
G: \text{VAR} \ \text{graph}[T]
\]

\[
G_{\text{pred}}(G): \text{TYPE} = P: \ \text{pred}[\text{graph}[T]] \mid (\exists X) (S: \ \text{graph}[T]): \text{subgraph}(S, G) \land P(S)
\]

We now define \( \text{max}_\text{subgraph} \) as follows:

\[
\text{max}_\text{subgraph}(G: \ \text{graph}[T], P: G_{\text{pred}}(G)): S: \ \text{Subgraph}(G) \mid \text{maximal}(G, S, P)
\]

The following useful properties of \( \text{max}_\text{subgraph} \) have been proved:

\[
\begin{align*}
\text{max}_\text{subgraph}_\text{def} & : \text{LEMMA} \ \forall (P: G_{\text{pred}}(G)): \text{maximal}(G, \text{max}_\text{subgraph}(G, P), P) \\
\text{max}_\text{subgraph}_\text{in} & : \text{LEMMA} \ \forall (P: G_{\text{pred}}(G)): P(\text{max}_\text{subgraph}(G, P)) \\
\text{max}_\text{subgraph}_\text{is}_\text{max} & : \text{LEMMA} \ \forall (P: G_{\text{pred}}(G)): \\
& \quad (\forall (SS: \ \text{Subgraph}(G)): P(SS) \implies \text{size}(SS) \leq \text{size}(\text{max}_\text{subgraph}(G, P)))
\end{align*}
\]

These definitions and lemmas are located in the theory \( \text{max}_\text{subgraphs} \).

A similar theory for subtrees is available in the theory \( \text{max}_\text{subtrees} \).

### A.4 Minimum Walks

Given that a walk \( w \) from vertex \( x \) to vertex \( y \) exists, we sometimes need to find the shortest walk from \( x \) to \( y \). The theory \( \text{min}_\text{walks} \) provides a function \( \text{min}_\text{walk}_\text{from} \) that returns a walk that is minimal. It is defined formally as follows:

\[
v1, v2, x, y: \text{VAR} \ T \\
G: \text{VAR} \ \text{graph}[T]
\]

\[
\text{gr}_\text{walk}(v1, v2): \text{TYPE} = G: \ \text{graph}[T] \mid \text{vert}(G)(v1) \land \text{vert}(G)(v2) \land \\
(\exists X) (w: \ \text{Seq}(G)): \text{walk}_\text{from}(G, w, v1, v2)
\]

\[
\text{min}_\text{walk}_\text{from}(x, y, (Gw: \text{gr}_\text{walk}(x, y))): \text{Walk}(Gw) = \\
\quad \text{min}[\text{Seq}(Gw), (\lambda w: \ \text{Seq}(Gw)): \ 1(w)], \\
\quad (\lambda w: \ \text{Seq}(Gw)): \text{walk}_\text{from}(Gw, w, x, y)]
\]

The following properties of \( \text{min}_\text{walk}_\text{from} \) have been established:
is_min(G,(w: Seq(G)),x,y): bool = walk?(G,w) AND 
   (FORALL (ww: Seq(G)): walk_from?(G,ww,x,y) IMPLIES 
    l(w) <= l(ww))

min_walk_def: LEMMA FORALL (Gw: gr_walk(x,y)):
   walk_from?(Gw,min_walk_from(x,y,Gw),x,y) AND 
   is_min(Gw, min_walk_from(x,y,Gw),x,y)

min_walk_in: LEMMA FORALL (Gw: gr_walk(x,y)):
   walk_from?(Gw,min_walk_from(x,y,Gw),x,y)

min_walk_is_min: LEMMA FORALL (Gw: gr_walk(x,y), ww: Seq(Gw)):
   walk_from?(Gw,ww,x,y) IMPLIES 
   l(min_walk_from(x,y,Gw)) <= l(ww)

reduced?(G: graph[T], w: Seq(G)): bool = 
   (FORALL (k: nat): k > 0 AND k < l(w) - 1 IMPLIES w(k-1) /= w(k+1))

x,y: VAR T
min_walk_is_reduced: LEMMA FORALL (Gw: gr_walk(x,y)):
   reduced?(Gw,min_walk_from(x,y,Gw))

These lemmas are available in the theories min_walks and min_walk_reduced.

A.5 Abstract Min and Max Theories

The need for a function that returns the smallest or largest object that satisfies a particular predicate arises in many contexts. For example, one may need a minimal walk from s to t or the maximal subgraph that contains a tree. Thus, it is useful to develop abstract min and max theories that can be instantiated in multiple ways to provide different min and max functions. Such a theory must be parameterized by

\[
\begin{align*}
T &: \text{TYPE} \\
\text{size}:&: [T \rightarrow \text{nat}] \\
P &: \text{pred}[T] \\
\end{align*}
\]

the type of the object for which a min function is needed
the “size” function by which objects are compared
the property that the min function must satisfy

Formally we have

abstract_min[T: TYPE, size: [T -> nat], P: pred[T]]: THEORY

and

abstract_max[T: TYPE, size: [T -> nat], P: pred[T]]: THEORY
To simplify the following discussion, only the `abstract_min` theory will be elaborated in detail. The `abstract_max` theory is conceptually identical.

In order for a minimum function to be defined, it is necessary that at least one object exists that satisfies the property. Thus, the theory contains the following assuming clause.

**ASSUMING**

\[ T_{ne}: \text{ASSUMPTION EXISTS } (t: T): P(t) \]

**ENDASSUMING**

Users of this theory are required to prove that this assumption holds for their type \( T \) (via PVS's TCC generation mechanism).

A function `minimal?`\((S: T)\)` is then defined as follows:

\[
\text{minimal?}(S): \text{bool} = P(S) \text{ AND } \left( \forall (SS: T): P(SS) \implies \text{size}(S) \leq \text{size}(SS) \right)
\]

Using PVS's dependent type mechanism, \( \text{min} \) is specified by constraining its return type to be the subset of \( T \) that satisfies `minimal?`:

\[
\text{min}: \{S: T \mid \text{minimal?}(S)\}
\]

If there are multiple instances of objects that are minimal, the theory does not specify which object is selected by \( \text{min} \). It just states that \( \text{min} \) will return one of the minimal ones. This definition causes PVS to generate the following proof obligation (i.e. TCC):

\[
\text{min_TCC1}: \text{OBLIGATION } (\exists x: S: T \mid \text{minimal?}(S)): \text{TRUE};
\]

This was proved using a function `min_f`, defined as follows:

\[
\text{is_one}(n): \text{bool} = (\exists (S: T): P(S) \text{ AND } \text{size}(S) = n)
\]

\[
\text{min_f}: \text{nat} = \text{min}_{[\text{nat}]}(n: \text{nat} \mid \text{is_one}(n))
\]

to construct the required \( \text{min} \) function. The \( T_{ne} \) assumption is sufficient to guarantee that \( \text{min_f} \) is well-defined.

The following properties have been proved about \( \text{min} \):

\[
\text{min_def}: \text{LEMMA } \text{minimal?}(\text{min})
\]

\[
\text{min_in} : \text{LEMMA } P(\text{min})
\]

\[
\text{min_is_min} : \text{LEMMA } P(SS) \implies \text{size}(\text{min}) \leq \text{size}(SS)
\]

These properties are sufficient for most applications.
This paper documents the NASA Langley PVS graph theory library. The library provides fundamental definitions for graphs, subgraphs, walks, paths, subgraphs generated by walks, trees, cycles, degree, separating sets, and four notions of connectedness. Theorems provided include Ramsey's and Menger's and the equivalence of all four notions of connectedness.